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On stability of Taylor vortices by fifth-order amplitude expansions

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Davey, Di Prima & Stuart's (1968) double amplitude expansion for disturbances in flow between concentric cylinders is formulated in matrix notation. The stability of the secondary equilibrium (Taylor-vortex) flow is calculated using fifth-order terms in amplitude, and using the full equations rather than the small-gap approximation. Qualitative confirmation is found of instabilities to the Taylor-vortex flow to non-axisymmetric disturbances at about 10 % above the first critical Taylor number.

1. Introduction

Consider two concentric circular cylinders, the outer one fixed, and the space between them filled with liquid. We define a Taylor number T proportional to the square of the angular velocity of the inner cylinder. At steady values of T below a certain critical value, T_c say, the flow is purely circumferential and is laminar Couette flow; at values of T above T_c, however, the flow is more complex and has a toroidal (or Taylor) vortex system superimposed on a modified circumferential flow. The vortices are spaced regularly along the axis with a definite periodicity, neighbours having opposite senses of rotation.

At still higher values of T, above another critical Taylor number T_1, the Taylor vortices are modified by a waviness in the azimuth and in fact become waves travelling in that direction. Both the number of complete waves and the new critical Taylor number T_1 are functions of the ratio η of the radii of the cylinders. Coles (1965), with $\eta = 0{\cdot}88$ found the value of T_1 to be about $1{\cdot}5T_c$ with 4 azimuthal waves. On the other hand Schwarz, Springett & Donnelly (1964), with $\eta = 0{\cdot}945$, found T_1 about $1{\cdot}05T_c$ with only one azimuthal wave.† At still higher speeds Coles observed a sequence of new equilibrium states.

Davey, Di Prima & Stuart (1968) (henceforth D. D. & S.) give a survey of the theoretical and experimental evidence and introduce a method of examining the interaction of certain axially-symmetric and non-axially-symmetric modes of disturbance of the basic Couette flow. This allows both an analysis of the stability of the Taylor-vortex flow and an examination of possible flows consequent upon an instability.

They found that, with $\eta = 0{\cdot}951$ and a Taylor number of about $1{\cdot}07T_c$, the

† We are interpreting their 'weak' non-axisymmetric mode as an instability of the Taylor vortices.

Taylor-vortex flow becomes unstable to a disturbance with one complete wave in the azimuth, although the preference for one wave is not very strong. The method is essentially valid only for Taylor numbers 'near' the first critical value T_c, so that Cole's experiments are probably outside the range of validity and were not considered.

D. D. & S.'s method involved expansion of the velocity in powers of an amplitude function of time, $A(t)$, up to $A^3(t)$, and the calculation of the instability of the Taylor vortices turned out to be a rather delicate matter. They felt that inclusion of fifth-order terms in $A(t)$ could possibly have a crucial effect. They also made the 'small-gap approximation'. This was known to be reasonable for linear stability theory, but its effect on the Taylor-vortex instability was not known.

The present work concentrates on the instability of the Taylor vortices for $\eta = 0{\cdot}951$, and was undertaken to find the effect of: (i) using fifth-order terms in the amplitude; (ii) using the full equations instead of the 'small-gap' equations.

The expansion procedure is reformulated in a matrix form, which allows more uniformity of treatment of the various ordinary differential equations which appear. This means that we need to do much less manipulation to make the problem suitable for computing.

It also turns out that the extension to a fifth-order expansion requires a very careful consideration of the method by which certain Landau constants and perturbation functions should be evaluated. This is because the Stuart–Watson expansion method does not uniquely determine the amplitude function $A(t)$.

Different choices of the Landau constants are possible leading to different determinations of $A(t)$ and to different perturbation functions, all of which are asymptotically equivalent as $T \to T_c$. We choose our constants in the same way as Reynolds & Potter (1967) for the channel flow problem, but discuss the consequences more fully. In particular we show that the present choice leads to a natural expansion in which $A(t)$ is the coefficient of the most unstable eigenfunction in a linear eigenfunction expansion.

Matkowsky (1970) has obtained a solution of a model problem similar to that of the development of the Taylor-vortex flow with time. He uses a systematic expansion of a disturbance to a steady-state solution in terms of a small parameter ϵ, where in our problem we would have $\epsilon^2 = (T - T_c)/T_c$. His amplitude function $A_1(t')$ for terms of order ϵ satisfies an equation

$$dA_1/dt' = 2A_1 - cA_1^3,$$

where $t' = \epsilon^2 t$, and no fifth-order terms appear at this stage. This means we must consider carefully how the present fifth-order expansions are meaningful, and we examine this question briefly in § 3. The conclusion we reach is that although fifth-order terms would not affect qualitatively the development of Taylor vortices, they could, nevertheless, be important in deciding the delicate matter of their stability.

We find that D. D. & S.'s results for the instability of the Taylor vortices (to non-axisymmetric disturbances out of phase by $\frac{1}{2}\pi$ in the axial direction with the Taylor vortices) are qualitatively confirmed. Details of the critical Taylor numbers are in table 4. We see that although the use of the full equations and of the fifth-order terms does change the critical values, the general picture of possible instabilities of the Taylor-vortex flow at values of T about 10 % above T_c is still apparent. Since the contribution of the fifth-order terms is fairly small it is probable that the present results are fairly accurate for the idealized case of infinite cylinder length considered here.

2. The non-dimensional equations

Let r, θ, z denote cylindrical polar co-ordinates. Consider two infinitely long concentric right circular cylinders with the z axis as their common axis. The inner and outer radii are R_1 and R_2, while the corresponding angular velocities are Ω_1 and Ω_2.

We start from the Navier–Stokes and continuity equations for viscous incompressible laminar flow. There is a basic steady Couette flow solution of the equations, though it is known to be stable only at Taylor numbers below a certain critical value T_c. The basic flow is

$$u_r = 0, \quad u_\theta = V(r) = Ar + B/r, \quad u_z = 0, \tag{2.1}$$

where
$$A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}, \quad B = \frac{(\Omega_1 - \Omega_2) R_1^2 R_2^2}{R_2^2 - R_1^2}$$

and the pressure is a known function p_0 of r only.

We set

$$u_r = u', \quad u_\theta = V(r) + v', \quad u_z = w', \quad p = p_0(r) + p',$$

where the primed variables are functions of r, θ, z and t, to obtain the disturbance equations satisfied by u', v', w' and p'. Since we wish to consider both the 'small-gap' and the full equations we choose our non-dimensional variables in the same way as D.D. & S. except for the definition of α below. First, we define the constants

$$R_0 = \tfrac{1}{2}(R_1 + R_2), \quad \Omega_0 = \tfrac{1}{2}(\Omega_1 + \Omega_2), \quad d = R_2 - R_1 \tag{2.2}$$

and the dimensionless constant

$$\alpha = -Ad/\Omega_0 R_0. \tag{2.3}$$

The dimensionless variables are x, ϕ, ζ, τ, and u, v, w, p, defined by

$$\left.\begin{aligned} r = R_0 + dx, \quad z = \zeta d, \quad \theta = (\Omega_0 d^2/\nu)\,\phi, \quad t = (d^2/\nu)\,\tau, \\ u' = -(\nu/\alpha d)\,u, \quad v' = \Omega_0 R_0 v, \quad w' = -(\nu/\alpha d)\,w, \quad p' = -(\nu^2\rho/\alpha d^2)\,p. \end{aligned}\right\} \tag{2.4}$$

Here ν is the kinematic viscosity and ρ the density. Using these variables we can express the equations of motion in a form containing just three independent dimensionless parameters

$$T = -4A\Omega_0 d^4/\nu^2, \quad \delta = d/R_0, \quad \mu = \Omega_2/\Omega_1. \tag{2.5}$$

The first of these, T, is called the Taylor number and we note that if $\Omega_2 = 0$ (as it will be in our calculations) then T is proportional to the square of Ω_1. We find it convenient notationally to use also the dimensionless parameter

$$\eta = R_1/R_2 \tag{2.6}$$

and to continue to use α (defined in (2.3)), noting that the two dimensionless parameters η and α are given in terms of δ and μ by using

$$\eta = \frac{2-\delta}{2+\delta}, \quad \alpha = \frac{8(\eta^2-\mu)}{(1+\mu)(1+\eta)^2}. \tag{2.7}$$

The equations of motion also contain the dimensionless functions

$$G(x) = 1/(1+\delta x) \tag{2.8}$$

and

$$\Omega_l(x) = \frac{2}{1+\mu}\frac{\mu-\eta^2}{1-\eta^2} + \frac{8\eta^2(1-\mu)}{(1+\eta)^2(1-\eta^2)(1+\mu)}G^2(x). \tag{2.9}$$

These are merely the expressions of R_0/r and $V(r)/(r\Omega_0)$ in terms of our dimensionless parameters and x.

Using the derivative with respect to x of the continuity equation to eliminate $\partial^2 u/\partial x^2$ from the first momentum equation we are able to write the equations as

$$\partial \mathbf{U}/\partial x - \mathbf{AU} - \mathbf{B}\,\partial \mathbf{U}/\partial \tau = \mathbf{L}(\mathbf{U})\,\mathbf{U}, \tag{2.10a}$$

where $\mathbf{AU}$ denotes the matrix product $\sum_{j=1}^{6} A_{ij}U_j$ and so on.

Here

$$\mathbf{U} = [u_0, v_0, w_0, u, v, w]^{\mathrm{Tr}} \tag{2.10b}$$

with Tr denoting the transpose;

$$\mathbf{A} = \begin{bmatrix} 0 & \alpha G\dfrac{\partial}{\partial\phi} & -\dfrac{\partial}{\partial\zeta} & \mathscr{B} & \alpha\delta G^2\dfrac{\partial}{\partial\phi} - T\Omega_l & 0 \\ -\dfrac{2\delta G}{T}\dfrac{\partial}{\partial\phi} & -\delta G & 0 & \dfrac{4\delta^2G^2}{T}\dfrac{\partial}{\partial\phi}+1 & -\mathscr{B}+\delta^2G^2 & 0 \\ \dfrac{\partial}{\partial\zeta} & 0 & -\delta G & 0 & 0 & -\mathscr{B} \\ 0 & 0 & 0 & -\delta G & \alpha G\dfrac{\partial}{\partial\phi} & -\dfrac{\partial}{\partial\zeta} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{2.10c}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} & \begin{matrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathscr{B} = \frac{\partial^2}{\partial\zeta^2} + \frac{2G^2\alpha\delta}{T}\frac{\partial^2}{\partial\phi^2} - \Omega_l\frac{\partial}{\partial\phi}, \tag{2.10d}$$

and

$$\mathbf{L}(\mathbf{U}) = -\frac{1}{\alpha}\begin{bmatrix} \mathbf{0} & \begin{matrix} \delta Gu + \alpha Gv\dfrac{\partial}{\partial\phi} - w\dfrac{\partial}{\partial\zeta} & -\alpha Gu\dfrac{\partial}{\partial\phi} + \alpha\dfrac{TG}{2}v & u\dfrac{\partial}{\partial\zeta} \\ v_0 + \delta Gv & -\alpha G\dfrac{\partial}{\partial\phi} + w\dfrac{\partial}{\partial\zeta} & 0 \\ w_0 & 0 & -\alpha Gv\dfrac{\partial}{\partial\phi} + w\dfrac{\partial}{\partial\zeta} \end{matrix} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{2.10e}$$

It is implied by the equations that v_0 and w_0, the second and third components of $\mathbf{U}$, are related to the fluid velocities v and w by

$$v_0 = \partial v/\partial x, \quad w_0 = \partial w/\partial x \tag{2.10f}$$

and that u_0 is the dimensionless pressure.

3. The expansion procedure

We first label boundary conditions as follows for a vector with six components.

β_1: the first three components to be zero at $x = \pm\frac{1}{2}$, (3.1)

β_2: the last three components to be zero at $x = \pm\frac{1}{2}$. (3.2)

The second set of conditions β_2 expresses the physical condition of zero disturbance velocity at the cylinders, while the first set β_1 is required later for certain adjoint problems.

Consider the linear problem

$$\partial \mathbf{U}/\partial x - \mathbf{A}\mathbf{U} - \mathbf{B}\,\partial \mathbf{U}/\partial \tau = 0;\ \beta_2. \tag{3.3}$$

A real solution of (3.3) is given by

$$\mathbf{U} = e^{a_0\tau}\mathbf{u}_1(x)\,e^{i\lambda\zeta} + e^{\tilde{a}_0\tau}\tilde{\mathbf{u}}_1(x)\,e^{-i\lambda\zeta} \tag{3.4}$$

(where λ is a real constant and a tilde denotes the complex conjugate) provided that a_0 is an eigenvalue and $\mathbf{u}_1$ is the corresponding eigenfunction of the eigenvalue problem for σ:

$$d\mathbf{u}/dx - \mathbf{A}^{(1,0)}\mathbf{u} - \sigma\mathbf{B}\mathbf{u} = 0;\ \beta_2. \tag{3.5}$$

Here the definition of $\mathbf{A}^{(p,q)}$ is

$$\mathbf{A}^{(p,q)}\left\{\begin{array}{c}\mathbf{A}^{(p,q)} \text{ is obtained from } \mathbf{A} \text{ by replacing} \\ \partial/\partial\zeta \text{ by } ip\lambda \text{ and } \partial/\partial\phi \text{ by } iqk.\end{array}\right\} \tag{3.6}$$

We note that $\mathbf{A}^{(p,q)}$ contains the parameters T, δ, μ, λ and k, and is a function of x. Problem (3.5) is just the usual linear stability problem of Couette flow for axisymmetric disturbances of wavelength $2\pi/\lambda$.

It is well known that only real eigenvalues are found, and here we suppose that a_0 is the greatest (most unstable) eigenvalue and that $\mathbf{u}_1$ is the corresponding eigenfunction. Computations, or previous work, show us that $\mathbf{u}_1(x)$ may be normalized to have real first, second, fourth and fifth components and purely imaginary third and sixth components. We regard $\mathbf{u}_1(x)$ as so normalized.

Another real solution of (3.3) is given by

$$\mathbf{U} = e^{b_0\tau}\mathbf{u}_2(x)\,e^{i\lambda\zeta+ik\phi} + e^{\tilde{b}_0\tau}\,\tilde{\mathbf{u}}_2(x)\,e^{-i\lambda\zeta-ik\phi}, \tag{3.7}$$

provided that b_0 is an eigenvalue and $\mathbf{u}_2$ the corresponding eigenfunction of the eigenvalue problem for

$$d\mathbf{u}/dx - \mathbf{A}^{(1,1)}\mathbf{u} - \gamma\mathbf{B}\mathbf{u} = 0; \quad \beta_2. \tag{3.8}$$

Since $\theta = (\Omega_0 d^2/\nu)\,\phi = \{T^{\frac{1}{2}}\delta^{\frac{1}{2}}/(2^{\frac{1}{2}}\alpha^{\frac{1}{2}})\}\,\phi$ the only admissable values of k are given by

$$k = m\{T\delta/2\alpha\}^{\frac{1}{2}}, \tag{3.9}$$

where m is an integer which specifies the number of complete waves in the azimuth. Here we suppose that b_0 is that eigenvalue with greatest real part of the non-axisymmetric linear stability problem (3.8). Computations tell us that both b_0 and $\mathbf{u}_2$ are complex.

A third real solution of (3.3) is given by

$$\mathbf{u} = e^{b_0\tau}\mathbf{v}_2(x)\, e^{-i\lambda\zeta+ik\phi} + e^{\tilde{b}_0\tau}\tilde{\mathbf{v}}_2(x)\, e^{i\lambda\zeta-ik\phi}, \tag{3.10}$$

provided that

$$d\mathbf{v}_2/dx - \mathbf{A}^{(-1,1)}\mathbf{v}_2 - b_0\mathbf{B}\mathbf{v}_2 = 0; \quad \beta_2. \tag{3.11}$$

It is easy to see by examination of $\mathbf{A}^{(1,1)}$ that if $\mathbf{u}_2$ is a solution of (3.8) then $\mathbf{v}_2 = K\hat{\mathbf{u}}_2$ is a solution of (3.11), where the notation $\hat{\mathbf{u}}$ means

$$\hat{\mathbf{u}} = \begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ 0 & & & & & -1 \end{bmatrix} \mathbf{u}, \tag{3.12}$$

and where K is any constant.

In order to take account of the non-linear terms $\mathbf{L}(\mathbf{U})\,\mathbf{U}$ in (2.10a) we shall expand the velocity $\mathbf{U}$ in a series beginning with expressions like (3.7) or (3.10), but with $e^{a_0\tau}$ replaced by a real amplitude function $F(\tau)$ and with $e^{b_0\tau}$ replaced by a complex amplitude function $H(\tau)$; and the linear relations $dF/d\tau = a_0F$ and $dH/d\tau = b_0H$ will be suitably modified. In other words we shall consider the interaction of certain basic linear eigenfunctions corresponding to the most unstable modes of linear theory.

Since we are hoping to demonstrate the instability of Taylor vortices, we are guided by D. D. & S. to consider the interaction of the linear eigenfunction (3.4) with that combination of (3.7) and (3.10) which is out of phase by $\frac{1}{2}\pi$ with (3.4) in the ζ direction. That combination which is in phase was shown to be stable by D. D. & S. and is not considered. Now (3.4) has a factor of $\cos\lambda\zeta$ in its first, second, fourth and fifth components, so we choose $\mathbf{v}_2 = -\hat{\mathbf{u}}_2$ in (3.11) and add (3.7) to (3.11) to obtain the appropriate combination which has a factor $\sin\lambda\zeta$ in its first, second, fourth and fifth components.

Thus we are led to an expansion of $\mathbf{U}$ in powers of $F(\tau)$, $H(\tau)$, $\tilde{H}(\tau)$:

$$\begin{aligned}\mathbf{U} = \mathbf{u}_{100}F(\tau) + \mathbf{u}_{010}H(\tau) + \mathbf{u}_{001}\tilde{H}(\tau) + \mathbf{u}_{200}F^2(\tau) + \mathbf{u}_{020}H^2(\tau) + \ldots \\ + \mathbf{u}_{210}F^2(\tau)\,H(\tau) + \ldots + \mathbf{u}_{lmn}F^l(\tau)\,H^m(\tau)\,\tilde{H}^n(\tau) + \ldots,\end{aligned} \tag{3.13}$$

where

$$\mathbf{u}_{100} = \mathbf{u}_1(x)\, e^{i\lambda\zeta} + \tilde{\mathbf{u}}_1(x)\, e^{-i\lambda\zeta}, \tag{3.14a}$$

$$\mathbf{u}_{010} = \mathbf{u}_2(x)\, e^{i\lambda\zeta+ik\phi} - \hat{\mathbf{u}}_2(x)\, e^{-i\lambda\zeta+ik\phi}, \tag{3.14b}$$

$$\mathbf{u}_{001} = \tilde{\mathbf{u}}_{010}. \tag{3.14c}$$

It turns out (though it is by no means clear at this stage) that in order to perform the expansion we must also set

$$dF/d\tau = a_0F + a_1F^3 + a_4FH\tilde{H} + a_5F^5 + a_7FH^2\tilde{H}^2 + \ldots, \tag{3.15}$$

$$dH/d\tau = b_0H + b_1H^2\tilde{H} + b_4F^2H + b_5F^4H + b_7H^3\tilde{H}^2 + \ldots, \tag{3.16}$$

where the constants $a_1, a_4, a_5, a_7, b_1, b_4, b_5, b_7$ can later be found by certain existence conditions at $a_0 = 0$ or $\mathrm{Re}\,(b_0) = 0$.

In partial explanation of (3.15) and (3.16) we note that if we use (3.13) for $\mathbf{U}$ we obtain

$$\begin{aligned}\partial\mathbf{U}/\partial\tau = {} & (\mathbf{u}_{100}+2F\mathbf{u}_{200}+2FH\mathbf{u}_{210}+3F^2\mathbf{u}_{300}+\ldots)\,(a_0F+a_1F^3+\ldots)\\ & +(\mathbf{u}_{010}+2H\mathbf{u}_{020}+\ldots)\,(b_0H+\ldots)\\ & +(\mathbf{u}_{001}+2\tilde{H}\mathbf{u}_{002}+\ldots)\,(\tilde{b}\tilde{H}_0+\ldots).\end{aligned}$$

Using this in (2.10*a*) we obtain, for example, that the coefficient of $F^3(\tau)$ gives

$$(\partial/\partial x-\mathbf{A}-3a_0\mathbf{B})\,\mathbf{u}_{300} = \mathbf{N}_{300}+a_1\mathbf{u}_{100}, \tag{3.16a}$$

where $\mathbf{N}_{300}$ is the non-linear contribution from earlier terms in the series.

Now when $a_0 = 0$ the homogeneous problem has a solution and the right-hand side must be adjusted, by choice of a_1, to ensure a solution. It looks at first sight as if we should need a term a_2F^2 in (3.15) as well, but upon expanding further in Fourier series it becomes clear that this is not needed. Similarly, the other terms in (3.15) and (3.16) are introduced to ensure solvability at $a_0 = 0$ or $\mathrm{Re}\,(b_0) = 0$.

For a problem like that of determining the development of the Taylor-vortex equilibrium flows ($H = 0$ for our expansions) Matkowsky (1970) performs a systematic expansion in terms of ϵ, where $\epsilon^2 = T-T_c$. By using a stretched time variable $t' = \epsilon^2\tau$ he is able to obtain an expansion starting with the term $\epsilon u_1(\mathbf{r})\,A_1(t')$, and shows by consideration of $O(\epsilon^3)$ terms that

$$dA_1/dt' = 2A-cA^3 \quad (c > 0).$$

No fifth-order terms are needed for the determination of $A_1(t')$. He also shows that higher-order amplitude functions $A_j(t')$ satisfy equations of the form

$$dA_j/dt'-(2-3cA_1^2)\,A_j = R_j$$

and thus argues that since $A_1 \rightarrow (2/c)^{\frac{1}{2}}$ then all the $A_j \rightarrow$ constant. D.D. & S. have given a different argument, using global stability analysis, that if the expansion up to $A^3(\tau)$ shows a tendency to a steady state, so will higher-order expansions.

If we use $t' = \epsilon^2\tau$ in (3.15) and set $H = 0$ with

$$F = \epsilon F_0(t')+\epsilon^2F_1(t')+\epsilon^3F_2(t')+\ldots$$

and

$$a_0 = k_1\epsilon^2+k_2\epsilon^3+\ldots,$$

we obtain, by equating powers of ϵ,

$$dF_0/dt' = k_1F_0+a_1F_0^3,$$

$$dF_1/dt'-(k_1+3a_1F_0^2)\,F_1 = k_2F_0$$

and so on, corresponding to Matkowsky's amplitude equations. But since $\mathrm{Re}\,(b_0)$ (although numerically small) does not tend to zero as $T \rightarrow T_c$ one could not use the same method to modify the equations (3.15) and (3.16) together, with $H \neq 0$.

By either our method or by Matkowsky's one can theoretically obtain the Taylor-vortex equilibrium flow to any order in a_0 or ϵ. To consider the stability of these flows we linearize in H, and calculate the linear growth rate, which appears in the form $c_1+\epsilon^2c_2+\epsilon^4c_3+\ldots$ and our fifth-order expansions allow us to find the $O(\epsilon^4)$ terms. These could be important in determining the zeros of the stability coefficient, since c_1 is numerically small. We should also point out that equations (3.15) and (3.16) allow a description of the development of steady wavy-vortex flow from the linear instability, though we do not pursue that in this paper.

Upon substitution of (3.13) in (2.10) we find that the forms of the spatial functions $\mathbf{u}_{lmn}$ for $l+m+n > 1$ are forced by the non-linear terms. If we pick out the coefficients of $F^lH^m\bar{H}^n\, e^{ip\lambda\zeta+iqk\phi}$ we obtain a set of ordinary differential equations which may be sequentially solved to give the flow field provided the constants in (3.15) and (3.16) are correctly chosen.

In our subsequent work we shall need the values of only the constants a_0, a_1, a_5, b_0, b_4 and b_5 and to obtain these values we need only the selection of equations set out below. The function $\mathbf{u}_j(x)$ on the left-hand side of the equation is in each case the coefficient of $F^lH^m\, e^{ip\lambda\zeta+iqk\phi}$ in the expansion of $\mathbf{U}$, as indicated. We define the operator $\mathscr{L}(p,q;K)$ by

$$\mathscr{L}(p,q;K)\,\mathbf{u} \equiv (d/dx-\mathbf{A}^{(p,q)}-K\mathbf{B})\,\mathbf{u}, \tag{3.17}$$

where $\mathbf{A}^{(p,q)}$ is defined in (3.6), and is a function of T, δ, μ, λ, m and x.

Then the set of ordinary differential equations is:

$$F\,e^{i\lambda\zeta}: \quad \mathscr{L}(1,0;\,a_0)\,\mathbf{u}_1 = 0, \tag{3.18}$$

$$H\,e^{i\lambda\zeta+ik\phi}: \quad \mathscr{L}(1,1;\,b_0)\,\mathbf{u}_2 = 0, \tag{3.19}$$

$$F^2\,e^{2i\lambda\zeta}: \quad \mathscr{L}(2,0;\,2a_0)\,\mathbf{u}_3 = \mathbf{N}_3, \tag{3.20}$$

$$F^2: \quad \mathscr{L}(0,0;\,2a_0)\,\mathbf{u}_4 = \mathbf{N}_4, \tag{3.21}$$

$$FH\,e^{2i\lambda\zeta+ik\phi}: \quad \mathscr{L}(2,1;\,a_0+b_0)\,\mathbf{u}_5 = \mathbf{N}_5, \tag{3.22}$$

$$FH\,e^{ik\phi}: \quad \mathscr{L}(0,1;\,a_0+b_0)\,\mathbf{u}_6 = \mathbf{N}_6, \tag{3.23}$$

$$F^3\,e^{3i\lambda\zeta}: \quad \mathscr{L}(3,0;\,3a_0)\,\mathbf{u}_7 = \mathbf{N}_7, \tag{3.24}$$

$$F^3\,e^{i\lambda\zeta}: \quad \mathscr{L}(1,0;\,3a_0)\,\mathbf{u}_8 = \mathbf{N}_8+a_1\mathbf{B}\mathbf{u}_1, \tag{3.25}$$

$$F^2H\,e^{3i\lambda\zeta+ik\phi}: \quad \mathscr{L}(3,1;\,2a_0+b_0)\,\mathbf{u}_9 = \mathbf{N}_9, \tag{3.26}$$

$$F^2H\,e^{i\lambda\zeta+ik\phi}: \quad \mathscr{L}(1,1;\,2a_0+b_0)\,\mathbf{u}_{10} = \mathbf{N}_{10}+b_4\mathbf{B}\mathbf{u}_2, \tag{3.27}$$

$$F^4\,e^{2i\lambda\zeta}: \quad \mathscr{L}(2,0;\,4a_0)\,\mathbf{u}_{11} = \mathbf{N}_{11}+2a_1\mathbf{B}\mathbf{u}_3, \tag{3.28}$$

$$F^4: \quad \mathscr{L}(0,0;\,4a_0)\,\mathbf{u}_{12} = \mathbf{N}_{12}+2a_1\mathbf{B}u_4, \tag{3.29}$$

$$F^3H\,e^{2i\lambda\zeta+ik\phi}: \quad \mathscr{L}(2,1;\,3a_0+b_0)\,\mathbf{u}_{13} = \mathbf{N}_{13}+(a_1+b_4)\,\mathbf{B}\mathbf{u}_5, \tag{3.30}$$

$$F^3H\,e^{ik\phi}: \quad \mathscr{L}(0,1;\,3a_0+b_0)\,\mathbf{u}_{14} = \mathbf{N}_{14}+(a_1+b_4)\,\mathbf{B}u_6, \tag{3.31}$$

$$F^5\,e^{i\lambda\zeta}: \quad \mathscr{L}(1,0;\,5a_0)\,\mathbf{u}_{15} = \mathbf{N}_{15}+3a_1\mathbf{B}\mathbf{u}_8+a_5\mathbf{B}\mathbf{u}_1, \tag{3.32}$$

$$F^4H\,e^{i\lambda\zeta+ik\phi}: \quad \mathscr{L}(1,1;\,4a_0+b_0)\,\mathbf{u}_{16} = \mathbf{N}_{16}+2a_1\mathbf{B}\mathbf{u}_{10}+b_5\mathbf{B}\mathbf{u}_2+b_4\mathbf{u}_{10}. \tag{3.33}$$

In every case we have the boundary conditions β_2: the last three components of $\mathbf{u}$ are to vanish at $x = \pm\frac{1}{2}$. Here the $\mathbf{N}_j$ are quadratic functions of vectors calculated from the $\mathbf{u}_n$ with $n < j$, and can be easily computed. Details are given in the appendix. The constants $a_1, \ldots, b_5$ are not known, and are found by applying solvability conditions to certain of the above equations, as described later.

4. The small-gap approximation

D. D. & S. performed their calculations using the 'small-gap approximation'. We will here merely describe this approximation in terms of the present notation.

First, in the partial differential equations (2.10) we let $\delta \to 0$, keeping T, μ and the independent variables x, ϕ and ζ fixed. The limiting value of $\Omega_l(x)$ is

$$\Omega_{l0}(x) = 1 - \frac{2(1-\mu)}{1+\mu}\,x \tag{4.1}$$

and we note also that $\alpha \to \alpha_0$ where

$$\alpha_0 = \frac{1-\mu}{1+\mu}. \tag{4.2}$$

We next perform the expansion procedure described in § 3, with values of k given by (3.9), where we do not set $\delta = 0$, but specify k by

$$k = m\{T\delta/2\alpha_0\}^{\frac{1}{2}} \quad (m = 0, 1, 2, \ldots). \tag{4.3}$$

For example, when we talk about the 'small-gap approximation with $\delta = 0{\cdot}05$' we mean that $\delta \to 0$ in the partial differential equations (2.10) but that values of k are chosen with $\delta = 0{\cdot}05$ in (4.3).

An alternative approach is to solve the system with arbitrary 'small' values of k (having first fixed T and μ) and to note that from (4.3)

$$\delta = 2\alpha_0 k^2/m^2 T, \tag{4.4}$$

so that any given value of k may be interpreted as applying to various values of δ, with $m = 1, 2, 3, \ldots$ in (4.4). This will obviously save computation time if we choose the values of T and k suitably and are able to interpolate accurately; but here we shall adopt the direct method of using (4.3) to specify k.

5. Method of calculation

5.1. *General aim*

We wish to investigate the stability of the Taylor-vortex equilibrium flow to small non-axisymmetric disturbances of the form (3.14*b*) plus its complex conjugate. Let us regard μ and δ as fixed; then T, the Taylor number, is the only free parameter in the basic equations (2.10). But when we consider the ordinary differential equations (3.18) to (3.33) there are two more wave-number parameters λ and m at our disposal. These specify the wavelength $2\pi/\lambda$ of the Taylor vortices and the number of azimuthal waves respectively. In doing any one calculation we shall keep λ and m fixed. The value of λ is, mathematically speaking, not determined but experiments indicate that Taylor vortices appear with $\lambda \simeq \lambda_c$, where λ_c is defined as follows. Consider the linear eigenvalue problem (3.5); let a_0 be the greatest of the eigenvalues. Draw the curve $a_0 = 0$ in the (λ, T) plane. Calculations show that this curve has a minimum, and we denote this minimum point by (λ_c, T_c). Above the curve $a_0 > 0$, below the curve $a_0 < 0$. In our calculations we keep λ fixed at $\lambda = \lambda_c$, or very close to λ_c, and use values of $T > T_c$.

We calculate the equilibrium amplitude F_e of the Taylor vortices by setting $dF/d\tau = 0$ and $H = 0$ in (3.15). This yields the equation

$$a_0 + a_1 F_e^2 + a_5 F_e^4 + \ldots = 0 \tag{5.1}$$

and we are interested in the root which has the asymptotic form (a_0 being real, a_1 real and negative, a_5 real)

$$F_e \sim (-a_0/a_1)^{\frac{1}{2}} \tag{5.2}$$

as $a_0 \rightarrow 0$. We suppose that a small amount of the non-axisymmetric disturbance is introduced; then the behaviour of the amplitude function $H(\tau)$ is given by (3.16) with $F = F_e$. Linearizing in H we find

$$dH/d\tau = (b_0 + b_4 F_e^2 + b_5 F_e^4 + \ldots) H. \tag{5.3}$$

Now F_e is real, so the quantity

$$b_{0r} + b_{4r} F_e^2 + b_{5r} F_e^4 + \ldots \tag{5.4}$$

is called the stability coefficient.† If it is positive the small disturbance will grow and if it is negative it will decay.

Our aim is to find the value of the stability coefficient in a form which is asymptotically correct with error $O(a_0^2)$ as $a_0 \rightarrow 0$ (i.e. as $T \rightarrow T_c$).

5.2. *On the linear eigenvalue problems and the adjoints*

Certain difficulties arise in evaluating the constants a_1, a_5, b_4 and b_5. To discuss these we need to make some preliminary remarks about the linear eigenvalue problems (3.5) and (3.8).

We assume that the linear eigenvalue problem (3.5) for axisymmetric disturbances of axial wavelength $2\pi/\lambda$ has a set of real distinct eigenvalues $\sigma_0, \sigma_1, \sigma_2, \ldots$, with corresponding eigenfunctions $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \ldots$. We order these so that σ_0 is the greatest, and remind the reader that a_0, $\mathbf{u}_1$ of equation (3.18) have been chosen to be σ_0, $\mathbf{e}_0$ respectively.

The adjoint eigenfunctions are the solutions of

$$d\mathbf{u}/dx + \{\mathbf{A}^{(1,0)}\}^{\mathrm{Tr}}\,\mathbf{u} + \sigma_n \mathbf{B}\mathbf{u} = 0; \quad \beta_1, \tag{5.5}$$

where the label β_1 describes the boundary conditions that the first three components of $\mathbf{u}$ are zero at $x = +\frac{1}{2}$ and at $x = -\frac{1}{2}$. We call the adjoint eigenfunctions $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2, \ldots$. Then in this notation the Fredholm alternative condition for the problem

$$d\mathbf{u}/dx - \mathbf{A}^{(1,0)}\mathbf{u} - \sigma_n \mathbf{B}\mathbf{u} = \mathbf{R}; \quad \beta_2 \tag{5.6}$$

to have a non-trivial solution is that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{f}_n^{\mathrm{Tr}}\,\mathbf{R}\,dx = 0. \tag{5.7}$$

We assume also that the linear eigenvalue problem (3.8) for non-axisymmetric disturbances of axial wavelength $2\pi/\lambda$, and with m azimuthal waves, has a set of

† The notation is that b_{0r} denotes the real part of b_0, and so on.

distinct complex eigenvalues $\gamma_0, \gamma_1, \gamma_2, \ldots$, where γ_0 has the largest real part. The corresponding eigenfunctions are called $\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \ldots$, while the adjoint eigenfunctions are called $\mathbf{h}_0, \mathbf{h}_1, \mathbf{h}_2, \ldots$. The condition for the problem

$$d\mathbf{u}/dx - \mathbf{A}^{(1,1)}\mathbf{u} - \gamma_n \mathbf{B}\mathbf{u} = \mathbf{Q}; \quad \beta_2 \tag{5.8}$$

to have a non-trivial solution is

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{h}_0^{\mathrm{Tr}} \mathbf{Q}\, dx = 0. \tag{5.9}$$

It is easy to show also that the orthogonality relations

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{f}_n^{\mathrm{Tr}} \mathbf{B} \mathbf{e}_m\, dx = \delta_{nm}, \tag{5.10}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{h}_n^{\mathrm{Tr}} \mathbf{B} \mathbf{g}_m\, dx = \delta_{nm} \tag{5.11}$$

are satisfied after suitable scaling of the functions.

5.3. *The solutions and the constants*

We now consider the set of ordinary differential equations (3.18) to (3.33). In § 3 we explained that a_0 and $\mathbf{u}_1$ of equation (3.18) are to be chosen as the greatest eigenvalue and corresponding eigenfunction of problem (3.5). We now fix δ, μ, λ (m does not appear) so that a_0 is a function of T only. It is convenient to regard a_0 as independent and think of T and $\mathbf{u}_1$ as functions of a_0.

We next select a value of m, specifying the number of complete azimuthal disturbance waves. We explained that b_0 and $\mathbf{u}_2$ of equation (3.19) are to be chosen as the most unstable eigenvalue and corresponding eigenfunction of (3.8). These, and all later functions and constants a_1, a_5, b_4, b_5 can be regarded as functions of a_0.

After solving for $\mathbf{u}_1$ and $\mathbf{u}_2$ we can easily calculate, at $a_0 = 0$ and values of $a_0 > 0$, $\mathbf{N}_3, \mathbf{N}_4, \mathbf{N}_5$ and $\mathbf{N}_6$ (see appendix) and proceed to solve for $\mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$ and $\mathbf{u}_6$ (see § 6 for further details). We can now calculate $\mathbf{N}_7$, $\mathbf{N}_8$, $\mathbf{N}_9$ and $\mathbf{N}_{10}$ and solve for $\mathbf{u}_7$ and $\mathbf{u}_9$.

But equations (3.25) and (3.27), for $\mathbf{u}_8$ and $\mathbf{u}_{10}$ respectively, do not have solutions when $a_0 = 0$ unless the constants a_1 and b_4 are specially chosen. This is because the corresponding homogeneous problems have eigenfunctions $\mathbf{u}_1$ and $\mathbf{u}_2$ when $a_0 = 0$. In order for solutions to exist when $a_0 = 0$ for equations (3.25) and (3.27) we must have

$$a_{10} = -\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{f}_0^{\mathrm{Tr}} \mathbf{N}_8\, dx\right]_{a_0=0}, \tag{5.12a}$$

$$b_{40} = -\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{h}_0^{\mathrm{Tr}} \mathbf{N}_{10}\, dx\right]_{a_0=0}, \tag{5.12b}$$

where a_{10}, b_{40} denote the values of a_1 and b_4 when $a_0 = 0$. Here we have used conditions (5.7) and (5.9) for existence of solutions.

Since we are interested in the value of the stability coefficient (5.4) when $a_0 > 0$ (i.e. when $T > T_c$) we must decide what values we should give to a_1 and b_4 when $a_0 > 0$; and how to go on to calculate the constants a_5 and b_5. These matters can be clarified by considering the solutions of equations (3.25) and (3.27) in more detail.

Let us suppose that, at general values of a_0, we expand $\mathbf{u}_8$ in terms of the eigenfunctions $\mathbf{e}_n$ of (3.5):

$$\mathbf{u}_8 = \sum_{n=0}^{\infty} c_n \mathbf{e}_n. \tag{5.13}$$

Both the c_n and the functions $\mathbf{e}_n$ are functions of a_0, and $\mathbf{u}_1 = \mathbf{e}_0$.

By substitution into (3.25), multiplication by $\mathbf{f}_m^{\mathrm{Tr}}$, the transpose of the mth adjoint eigenfunction, and integration with respect to x over the interval we obtain with the use of (5.6) and (5.10)

$$2\sigma_m c_m = -\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{f}_m^{\mathrm{Tr}} \mathbf{N}_8 \, dx \quad (m > 0), \tag{5.14}$$

$$2a_0 c_0 = -\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{f}_0^{\mathrm{Tr}} \mathbf{N}_8 \, dx - a_1. \tag{5.15}$$

Remember here that $\mathbf{N}_8$, $\mathbf{f}_m^{\mathrm{Tr}}$, $\mathbf{f}_0^{\mathrm{Tr}}$ are all functions of a_0. Thus in order for c_0 to be finite when $a_0 = 0$ we must have condition (5.12*a*), as already noted from a different point of view. At values of $a_0 > 0$ we see that the choice of a_1 determines the multiple of $\mathbf{e}_0$ occurring in $\mathbf{u}_8$. It seems reasonable to restrict ourselves to values of a_1 and c_0 which have derivatives with respect to a_0 at $a_0 = 0$, so we now write

$$\left.\begin{aligned} c_0 &= c_{00} + a_0 c_{11} + \ldots, \\ a_1 &= a_{10} + a_0 a_{11} + \ldots, \\ I \equiv \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{f}_0^{\mathrm{Tr}} \mathbf{N}_8 \, dx &= I_0 + a_0 I_1 + \ldots. \end{aligned}\right\} \tag{5.16}$$

We are implicity assuming here that $\mathbf{f}_0$ and $\mathbf{N}_8$ are also differentiable with respect to a_0 at $a_0 = 0$. There is no need to expand the eigenfunctions $\mathbf{e}_n$, though they can in principle be expanded in the same way.

By substitution into (5.15) and equating powers of a_0 we see that

$$a_{10} = -I_0, \tag{5.17}$$

and

$$2c_{00} = -a_{11} - I_1. \tag{5.18}$$

Equation (5.17) has already been noted as equation (5.12*a*). Now c_{00} is the coefficient of $\mathbf{e}_0 (= \mathbf{u}_1)$ in $\mathbf{u}_8$ when $a_0 = 0$, so we see that the choice of a_{11} determines the multiple of $\mathbf{e}_0$ which we must include in $\mathbf{u}_8$ when we solve (3.25) at $a_0 = 0$.

Similarly, if we expand $\mathbf{u}_{10}$ in terms of the eigenfunctions $\mathbf{g}_n$ of (3.8), (reminding the reader that $\mathbf{u}_2 = \mathbf{g}_0$):

$$\mathbf{u}_{10} = \sum_{n=0}^{\infty} d_n \mathbf{g}_n \tag{5.19}$$

and use the differential equation (3.27), the differential equation for the eigenfunction (3.8) and the orthogonality relations (5.11) we find, for all values of a_0,

$$2a_0 d_0 = -\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{h}_0^{\mathrm{Tr}} \mathbf{N}_{10} \, dx - b_4. \tag{5.20}$$

If we now write

$$\left.\begin{aligned} b_4 &= b_{40} + a_0 b_{41} + \ldots, \\ d_0 &= d_{00} + a_0 d_{01} + \ldots, \\ J \equiv -\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{h}_0^{\mathrm{Tr}} \mathbf{N}_{10}\, dx &= J_0 + a_0 J_1 + \ldots, \end{aligned}\right\} \tag{5.21}$$

we find from (5.20) that

$$b_{40} = -J_0 \quad \text{and} \quad 2d_{00} = -b_{41} - J_1. \tag{5.22}$$

The first of these equations (5.22) has already been noted as equation (5.12*b*), and we see from the second that the choice of b_{41} determines the multiple d_{00} of the eigenfunction $\mathbf{g}_0\ (=\mathbf{u}_2)$ occurring in $\mathbf{u}_{10}$ at $a_0 = 0$, (under the assumption of differentiability with respect to a_0 at $a_0 = 0$ of $\mathbf{u}_{10}$, $\mathbf{N}_{10}$, etc.).

As far as obtaining an asymptotically correct expression for the stability coefficient (5.4) is concerned, *any* finite values of a_{1n} and b_{4n} for $n > 0$ are allowable; for changes are induced in a_5 and b_5 (through the functions $\mathbf{u}_8$, $\mathbf{u}_{10}$ and subsequent equations) in a complicated way such that $b_{0r} + b_{4r}F_e^2 + b_{5r}F_e^4$ remains a correct asymptotic approximation to the stability coefficient with error $o(a_0^2)$ as $a_0 \to 0$. From this point of view all choices are equally valid; but nevertheless different choices will lead to different constants K in the error which is presumably asymptotic to Ka_0^3, that is to different amounts of higher-order terms. This lack of uniqueness in our representation of the physical system arises from a certain lack of precision in our definitions of $F(\tau)$ and $H(\tau)$.

The ambiguity disappears if we define $F(\tau)$ and $H(\tau)$ more precisely in a natural way as follows. The function $F(\tau)$ is defined to be the coefficient, in the velocity vector $\mathbf{U}(x, \phi, \zeta, \tau)$, of the spatial function $\mathbf{e}_0(x)\, e^{i\lambda\zeta}$ and $H(\tau)$ is defined to be the coefficient of $\mathbf{g}_0(x)\, e^{i\lambda\zeta + ik\phi}$. These spatial functions are in a sense eigenfunctions of the linear problem for $\mathbf{U}$, and we could more generally expand $\mathbf{U}$ in eigenfunctions of the linear problem:

$$\mathbf{U}(x, \phi, \zeta, \tau) = \mathrm{Re} \sum_{n,p,q=0}^{\infty} F_n^{(p,q)}(\tau)\, \mathbf{u}_n^{(p,q)}(x)\, e^{ip\lambda\zeta + iqk\phi},$$

where $\mathbf{u}_n^{(1,0)} = \mathbf{e}_n$ and $\mathbf{u}_n^{(1,1)} = \mathbf{g}_{0n}$. We are merely identifying $F(\tau)$ with $F^{(1,0)}(\tau)$ and $H(\tau)$ with $F^{(1,1)}(\tau)$.

Now since, in the expansion of $\mathbf{U}$, $\mathbf{u}_8$ is multiplied by $F^3(\tau)\, e^{i\lambda\zeta}$ we must clearly choose a_1 such that $\mathbf{u}_8$ contains no multiple of the eigenfunction $\mathbf{e}_0$; that is we must choose $c_0 = 0$ in (5.13) both when $a_0 = 0$ and when $a_0 > 0$. For otherwise the coefficient of $\mathbf{e}_0(x)\, e^{i\lambda\zeta}$ is $F(\tau) + c_0 F^3(\tau) + \ldots$ contrary to our definition of $F(\tau)$. So referring to (5.15) we see that, at all values of a_0, we should choose

$$a_1 = -\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{f}_0^{\mathrm{Tr}} \mathbf{N}_8\, dx. \tag{5.23}$$

Similarly, we must choose b_4 in such a way that $\mathbf{u}_{10}$ contains no multiple of the eigenfunction $\mathbf{g}_0$. Thus, from (5.20), at all values of a_0,

$$b_4 = -\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{h}_0^{\mathrm{Tr}} \mathbf{N}_{10}\, dx. \tag{5.24}$$

Equations (5.12*a*) and (5.12*b*) are then satisfied. But in order to actually calculate $\mathbf{u}_8$ and $\mathbf{u}_{10}$ at $a_0 = 0$ we must remember that the differential equations do not determine the functions uniquely,† and we must be careful to choose those solutions which contain no multiple of the eigenfunctions $\mathbf{e}_0$ and $\mathbf{g}_0$ respectively. We explain in § 6 how this is achieved.

Concerning the values of a_5 and b_5 we note that they are, by the same arguments as above, uniquely determined provided that we define $F(\tau)$ and $H(\tau)$ as we have done. But to obtain a solution for the stability coefficient correct to order a_0^2, we note that $F_e^2 \sim a_0$ so that we need only find the values a_{50} and b_{50} of a_5 and b_5 at $a_0 = 0$.

This is achieved by solving equations (3.24) onwards only at $a_0 = 0$, and using (5.7) and (5.9) as conditions for the existence of solutions of (3.32) and (3.33) at $a_0 = 0$. This gives the values of a_5 and b_5 at $a_0 = 0$ to be

$$a_{50} = \left[-\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{f}_0^{\mathrm{Tr}} \mathbf{N}_{15}\, dx - 3a_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{f}_0^{\mathrm{Tr}} \mathbf{B} \mathbf{u}_8\, dx \right]_{a_0=0}, \tag{5.25}$$

$$b_{50} = \left[-\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{h}_0^{\mathrm{Tr}} \mathbf{N}_{16}\, dx - 2a_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{h}_0^{\mathrm{Tr}} \mathbf{B} \mathbf{u}_{10}\, dx - b_4 \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{h}_0^{\mathrm{Tr}} \mathbf{B} \mathbf{u}_{10}\, dx \right]_{a_0=0}. \tag{5.26}$$

Finally, to find the stability coefficient (5.3) as a function of a_0, we use b_0 as the function of a_0 obtained from the eigenvalue problem (3.5), use a_1 as a function of a_0 from (5.23), and b_4 from (5.24), but use $b_5 = b_{50}$ and $a_5 = a_{50}$. This will give the value of the stability coefficient with error presumably $O(a_0^3)$ as $a_0 \to 0$.

In terms of the parameter T, this means that we choose T, solve for a_0, b_0, a_1, b_4 as functions of T, but evaluate a_5 and b_5 only at $T = T_c$, and our result for the stability coefficient will be an asymptotic approximation as $T \to T_c$.

6. The numerical work

To solve the eigenvalue problem (3.18) we followed the method used by previous authors, for example see Krueger, Gross & Di Prima (1966). That is, by means of a fourth-order Runge–Kutta scheme three independent integrals of the differential equation were found, each satisfying the boundary conditions at $x = -\frac{1}{2}$. These were called $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$ with initial values $V_{ij} = \delta_{ij}$, where V_{ij} denotes the jth component of $\mathbf{V}_i$. If $A\mathbf{V}_1 + B\mathbf{V}_2 + C\mathbf{V}_3$ is the eigenfunction then at $x = \frac{1}{2}$

$$|V_{ij}| = 0 \quad (i = 1, 2, 3; j = 4, 5, 6). \tag{6.1}$$

The zero of this determinant as a function of a_0 (for fixed values of T, μ, δ, λ) was located by a root-finding routine using linear interpolation. It was then easy to find the eigenfunction $\mathbf{u}_1$ by solving for $A{:}B{:}C$. Complex arithmetic and the matrix formulation were used, so that the same programme could then be used to calculate b_0 and $\mathbf{u}_2$ of equation (3.19).

† For $a_0 > 0$ the solutions for $\mathbf{u}_8$ and $\mathbf{u}_{10}$ are uniquely determined by our choice of a_1 and b_4. We are merely enforcing continuity conditions on $\mathbf{u}_8$ and $\mathbf{u}_{10}$ as functions of a_0, at $a_0 = 0$, by making the proper definitions at $a_0 = 0$.

Next we found the adjoint eigenfunctions $\mathbf{f}_0$ and $\mathbf{h}_0$, and appropriate normalizations were made to $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{f}_0$ and $\mathbf{h}_0$ to satisfy (5.8) and (5.11). It was found important not to scale $\mathbf{u}_1$ or $\mathbf{u}_2$ to be too large or too small, otherwise very large or small numbers appeared in the right-hand sides of later equations. In our final calculations we took the second component of both $\mathbf{u}_1$ and $\mathbf{u}_2$ to equal 1·0 at $x = -\frac{1}{2}$.

By using 20 Runge–Kutta steps we obtained, at worst, agreement to within 2 or 3 units in the fourth figure with D. D. & S.'s calculations for the eigenvalues in the small-gap approximation, and with Krueger *et al.*'s calculations for the full equations.

We then proceeded to problems (3.20) to (3.23). Particular care was needed in finding the function $\mathbf{u}_4$, and in finding $\mathbf{u}_6$ in the small-gap approximation. This was because for these problems the corresponding homogeneous problems *do* have eigenfunctions, with our present formulation, both when $a_0 = 0$ and when $a_0 > 0$. These eigenfunctions have the first component (pressure) constant and all other components zero, as can be seen by inspection of $\mathbf{A}^{(1,0)}$ and $\mathbf{A}^{(0,0)}$, and may be designated as 'almost trivial'. In D.D. & S.'s formulation the corresponding ordinary scalar differential equations for v do *not* have eigenfunctions. However, the orthogonality condition for existence of solutions was easily seen to be satisfied, so we went on to solve problems (3.20) to (3.23) by means of a routine written to deal with the non-homogeneous problem in general.

For $\mathbf{u}_n$, $n > 2$, we used the following method. We first integrated the homogeneous forms

$$d\mathbf{u}_n/dx - \mathbf{A}^{(p,q)}\mathbf{u}_n - K\mathbf{B}\mathbf{u}_n = 0 \tag{6.2}$$

with initial values, [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0] and [0, 0, 1, 0, 0, 0] to obtain the functions $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$. Then the right-hand sides were inserted. This involved only the repeated use of a routine to evaluate the vector function

$$\mathbf{R}(p, q, p', q', \mathbf{u}_l, \mathbf{u}_m)$$

described in the appendix. The equation was then integrated (after interpolation to half-step points of the right-hand side for use in the Runge–Kutta scheme) to find a particular integral $\mathbf{P}$ with initial values [1, 1, 1, 0, 0, 0). To find a solution of the form $A\mathbf{W}_1 + B\mathbf{W}_2 + C\mathbf{W}_3 + \mathbf{P}$ we needed

$$\left.\begin{aligned} AW_{14} + BW_{24} + CW_{34} + P_4 &= 0 \\ AW_{15} + BW_{25} + CW_{35} + P_5 &= 0 \\ AW_{16} + BW_{26} + CW_{36} + P_6 &= 0 \end{aligned}\right\} \quad \text{at} \quad x = \tfrac{1}{2}. \tag{6.3}$$

If no eigenfunction of the homogeneous problem exists then $|W_{ij}| \neq 0$ ($i = 1, 2, 3; j = 4, 5, 6$), so we may solve directly for A, B, and C. However, if an eigenfunction does exist then there is not a unique solution for A, B, C. We may set (in general) one of the constants arbitrarily and use just two of the equations to find the others. For example, if we set $A = 1$ and use the first two equations we can find B and C provided that $W_{24}W_{35} - W_{34}W_{25} \neq 0$. Therefore we wrote a routine to deal with this case which used the largest 2×2 minor for the divisor in calculating the constants. The solution obtained in this way contained a multiple of the eigenfunction, but for the functions $\mathbf{u}_4$ and $\mathbf{u}_6$ those eigenfunctions consisted merely of a constant first component, and it could be seen by examination of the $\mathbf{N}_j$ that this component did not affect any of the later functions. Since the first component is pressure, this is in accordance with the physical situation.

We thus solved for $\mathbf{u}_3$ to $\mathbf{u}_6$, and then computed a_1 and b_4 by means of (5.23) and (5.24). These calculations were done with fixed μ, δ, λ and m both at $a_0 = 0$ ($T = T_c$) and at values of $a_0 > 0$ ($T > T_c$).

We compared our results with those of D. D. & S. for the small-gap approximation with $\mu = 0$, $\delta = 0{\cdot}05$, $\lambda = 3{\cdot}127$, $m = 1$, $a_0 = 0$. Having access, through the courtesy of the authors, to some detailed figures, we were able to compare many of the components of our vector functions $\mathbf{u}_j$ with the corresponding functions of D. D. & S. We found agreement to about 4 figures in all cases except that of the function $\mathbf{u}_5$, where the discrepancy for one component was about 2 % of its absolute value at typical points. We checked our equations for this component against D. D. & S.'s and found (after some labour) that they were equivalent. Therefore the error lies somewhere in the computing methods or in the programming. We cannot say with complete certainty whose results are correct. Nevertheless, since we used a uniform method for solving all our equations and found agreement with many functions of D. D. & S., and checked our programme extensively we believe the present results to be correct. In any case, the discrepancy leads only to a difference of about 2 % in the imaginary part of b_4, the real part not being affected. Tests were made by changing the imaginary part of b_4 arbitrarily by this amount and seeing the effect on the later constant b_5. It was found to be negligible.

We also compared our result for a_1 at $a_0 = 0$ with the result of Davey (1962) for the full equations with $\mu = 0$, $\delta = \frac{2}{3}$, $\lambda = 3{\cdot}163$ and found a difference of less than 0·1 %.

We then went on to solve for $\mathbf{u}_8$ and $\mathbf{u}_{10}$ at $a_0 = 0$ only. We recall that $\mathbf{u}_8$ and $\mathbf{u}_{10}$ are not uniquely defined at $a_0 = 0$ by problems (3·25) and (3·27), but that in accordance with our choice of a_1 and b_4 we need those solutions which contain no multiples of the eigenfunctions $\mathbf{e}_0$ $(= \mathbf{u}_1)$ and $\mathbf{g}_0$ $(= \mathbf{u}_2)$. This was easily achieved as follows. Consider the case of $\mathbf{u}_8$. We first calculated a general solution u_{8G}, containing an unknown multiple of the eigenfunction $\mathbf{u}_1$, i.e. containing an unknown value of c_0 in expansion (5.13), at $a_0 = 0$. Thus

$$\mathbf{u}_{8G} = \sum_{n=0}^{\infty} c_n \mathbf{e}_n,$$

and by using the orthogonality condition (5.10) we can show that

$$c_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{f}_0^{\mathrm{Tr}} \mathbf{B} \mathbf{u}_{8G}\, dx.$$

We therefore subtracted $c_0 \mathbf{u}_1$ from $\mathbf{u}_{8G}$ to obtain $\mathbf{u}_8$, all these calculations being performed at $a_0 = 0$. After following a similar procedure for $\mathbf{u}_{10}$ our final functions $\mathbf{u}_8$ and $\mathbf{u}_{10}$ satisfied

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{f}_0^{\mathrm{Tr}} \mathbf{B} \mathbf{u}_8\, dx = 0, \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{h}_0^{\mathrm{Tr}} \mathbf{B} \mathbf{u}_{10}\, dx = 0 \quad \text{at} \quad a_0 = 0. \tag{6.4}$$

We then went on to solve for the functions $\mathbf{u}_{11}$ to $\mathbf{u}_{14}$, at $a_0 = 0$ only. We had to remember that in the case of $\mathbf{u}_{12}$, and in the case of $\mathbf{u}_{14}$ with the small-gap approximation, the corresponding homogeneous problems did have the 'nearly trivial' eigenfunctions mentioned earlier, and to use the appropriate method of solution.

We then calculated a_5 and b_5 at $a_0 = 0$ by formulae (5.25) and (5.26); and calculated the equilibrium amplitudes F_e and the stability coefficient at various values of a_0 (i.e. of T) as described earlier.

We performed tests with 10, 14, 20 and 40 steps for some of the cases and found that the error in each of the constants appeared to be closely proportional to the fourth power of the step-length. Figures given in the tables are those obtained using 20 steps. Typically the values for the constants and for F_e are estimated to have errors of less than 0·05 %. However, the stability coefficients could not be calculated to within such a good percentage error, because they occur as differences between expressions involving the constants in such a way that we lose some accuracy. The results for the $m = 1$ mode are believed to be accurate to within ± 2 in the last figure quoted. For $m = 2$ and 4 we have quoted less figures, and these are believed to be correct. Here we are talking about the accuracy as a solution of the defined mathematical procedure, and are not attempting a discussion of the error in the asymptotic series.

7. The results

It was decided to do two cases. The first was the small-gap approximation with $\delta = 0{\cdot}05$ and $\mu = 0$. The main object of this was to compare the present results with those of D.D. & S. to find the effect of including terms in $F^5(\tau)$ in the expansion of the velocity, or equivalently of including terms of order a_0^2 in the stability coefficient.

The second case was one using the full equations, again with $\delta = 0{\cdot}05$ and $\mu = 0$. In both cases we took $\lambda = 3{\cdot}127$. This is an accurate value for λ_c in the first case, and is close to λ_c in the second case. We kept the same λ in order to have a direct comparison of the results of the small-gap and the full equations with all the parameters the same.

The results are given in tables 1 to 4. One point should be emphasized. The values of the constants a_1, a_5, F_e, b_4 and b_5 are not absolute, but depend upon the scaling chosen for eigenfunction $\mathbf{u}_1$. If $\mathbf{u}_1$ is multiplied by any constant K (which must be real by the nature of our expansion), then a_1, b_4 are multiplied by K^2, F_e^2 is multiplied by $1/K^2$, and a_5, b_5 are multiplied by K^4. The scaling adopted here was that the second component of $\mathbf{u}_1$ was made equal to 1·0 at $x = -\frac{1}{2}$. The scaling for $\mathbf{u}_2$ does not affect any of our results. The eigenvalues a_0, b_0 and the stability coefficient $b_{0r} + b_{4r}F_e^2 + b_{5r}F_e^4$ are not affected by the scaling of $\mathbf{u}_1$.

In § 5.4 we explained the method of calculation, using a_0 as our descriptive parameter for convenience. Here we have used the Taylor number as our specified parameter, a_0 being calculated by first specifying T and then solving the eigenvalue problem (3.5) for the largest a_0. We chose to calculate those values of a_1 and b_4 which are given by (5.23) and (5.24) at each value of T; and a_{50} and b_{50}

(*a*) Small-gap, $\mu = 0$, $\delta = 0{\cdot}05$, $\lambda = 3{\cdot}127$

T	a_0	a_1	a_{50}	F_e^2
1695	0·0000	−4·892	−2·852	0
1735	0·3077	−4·983	—	0·0597
1775	0·6120	−5·071	—	0·1134
1815	0·9132	−5·159	—	0·1624
1855	1·2115	−5·245	—	0·2076

(*b*) Full equations, $\mu = 0$, $\delta = 0{\cdot}05$, $\lambda = 3{\cdot}127$

T	a_0	a_1	a_{50}	F_e^2
1753	0·0000	−5·533	−3·370	0
1843	0·6619	−5·737	—	0·1085
1903	1·0972	−5·871	—	0·1702
1933	1·3126	−5·937	—	0·1987
1963	1·5266	−6·002	—	0·2257
1993	1·7390	−6·068	—	0·2515

TABLE 1. Taylor-vortex constants

(*a*) $m = 1$. Here $b_{50} = -2{\cdot}590 - i0{\cdot}0833$

					Stability coeff.	
T	b_{0r}	b_{0i}	b_{0i}	b_{4i}	D.D. & S.	Present
1695	−0·0477	−4·844	−4·630	0·3600	−0·0477	−0·0477
1735	0·2590	−4·903	−4·735	0·3661	−0·0325	−0·0330
1775	0·5624	−4·962	−4·833	0·3724	−0·0174	−0·0192
1815	0·8628	−5·019	−4·928	0·3789	−0·0023	−0·0060
1855	1·1602	−5·077	−5·022	0·3855	+0·0126	+0·0063

(*b*) $m = 2$. Here $b_{50} = -1{\cdot}875 - i0{\cdot}1392$

					Stability coeff.	
T	b_{0r}	b_{0i}	b_{4r}	b_{4i}	D.D. & S.	Present
1695	−0·1917	−9·690	−3·878	0·7607	−0·192	−0·192
1735	+0·1126	−9·808	−4·009	0·7700	−0·131	−0·134
1775	0·4135	−9·925	−4·133	0·7804	−0·071	−0·080
1815	0·7114	−10·041	−4·252	0·7912	−0·013	−0·029
1855	1·0064	−10·155	−4·365	0·8028	+0·046	+0·019

(*c*) $m = 4$. Here $b_{50} = +0{\cdot}1822 - i0{\cdot}1671$

					Stability coeff.	
T	b_{0r}	b_{0i}	b_{4r}	b_{4i}	D.D. & S.	Present
1695	−0·7679	−19·396	−1·103	1·803	−0·768	−0·768
1735	−0·4737	−19·633	−1·335	1·803	−0·543	−0·553
1775	−0·1829	−19·867	−1·548	1·809	−0·321	−0·356
1815	+0·1050	−20·099	−1·748	1·818	−0·101	−0·174
1855	+0·3900	−20·328	−1·933	1·829	+0·116	−0·003

TABLE 2. Constants and stability coefficients for disturbances with m aximuthal waves. Small-gap, $\mu = 0$, $\delta = 0{\cdot}05$, $\lambda = 3{\cdot}127$, ($T_c = 1695$)

are the values of a_5 and b_5 at $T = T_c$ ($a_0 = 0$) consistent with our choices of a_1 and b_4. (If we had chosen $a_1 = a_{10}$ at all values of T, then we should have had a different value of a_{50}, since this depends, in a complicated way, on the value of da_1/da_0 at $a_0 = 0$.)

(a) $m = 1$. Here $b_{50} = -3{\cdot}072 - i0{\cdot}0394$

T	b_{0r}	b_{0i}	b_{4r}	b_{4i}	Stability coeff.
1753	−0·0555	−4·9263	−5·270	0·4607	−0·0555
1843	+0·6044	−5·0556	−5·496	0·4739	−0·0278
1903	1·0384	−5·1405	−5·641	0·4835	−0·0109
1963	1·4664	−5·2241	−5·788	0·4937	+0·0037

(b) $m = 2$. Here $b_{50} = -2{\cdot}257 - i0{\cdot}0156$

T	b_{0r}	b_{0i}	b_{4r}	b_{4i}	Stability coeff.
1753	−0·2220	−9·8545	−4·503	0·9655	−0·222
1903	+0·8618	−10·2831	−4·968	1·002	−0·049
1933	1·0745	−10·3671	−5·054	1·010	−0·019
1963	1·2858	−10·4504	−5·137	1·017	+0·011

(c) $m = 4$. Here $b_{50} = 0{\cdot}1308 + i0{\cdot}2595$

T	b_{0r}	b_{0i}	b_{4r}	b_{4i}	Stability coeff.
1753	−0·8896	−19·725	−1·741	2·229	−0·890
1933	+0·3585	−20·752	−2·656	2·248	−0·164
1963	0·5618	−20·919	−2·786	2·257	−0·060
1993	0·7637	−21·085	−2·911	2·268	+0·040

TABLE 3. Constants and stability coefficients for disturbances with m azimuthal waves. Full equations, $\mu = 0$, $\delta = 0{\cdot}05$, $\lambda = 3{\cdot}127$ ($T_c = 1753$)

Here $\mu = 0$, $\delta = 0{\cdot}05$, $\lambda = 3{\cdot}127$. The figures in brackets are percentages above T_c.

	D.D. & S. (small-gap)	Present calcs. (small-gap)	Present calcs. (full equations)
T_c	1695	1695	1753
$m = 1$	1820 (7 %)	1835 (8 %)	1946 (11 %)
$m = 2$	1824 (8 %)	1838 (8 %)	1951 (11 %)
$m = 4$	1837 (8 %)	1855 (9 %)	1981 (12 %)

TABLE 4. The critical Taylor numbers for the onset of instability of Taylor-vortex flow to modes with m azimuthal waves

Although we have talked of a_0 as a small parameter the reader will notice that our results extend to values of a_0 greater than unity. By comparison of our fifth-order expansion results with the third-order expansion results of D. D. & S. we conclude that the results are probably meaningful. Mathematically speaking we could for example just say that $0{\cdot}1a_0$ is our small parameter; presumably there is some other more natural small parameter related to a_0 which we could use, but there is no obvious choice.

To calculate the stability coefficients we used the expression $b_0 + b_{4r}F_e^2 + b_{50r}F_e^4$, where F_e is given by $a_0 + a_1F_e^2 + a_{50}F_e^4 = 0$. D. D. & S. used $b_{0r} + b_{40r}A_e^2$ where A_e is given by $a_0 + a_{10} A_e^2 = 0$. They expressed some serious doubts about the validity of their answers. We quote: "The neglected terms, due to the truncation of the amplitude equations at cubic terms, may well be smaller than each of the terms b_{0r} and $b_{4r} a_0/a_1$ individually, but may be very important in the neighbourhood of the zero of that coefficient. Indeed it is conceivable that such terms could even prevent the occurrence of a zero ...". However, after expressing these doubts D. D. & S. then went on to give a rough estimate of the error in the critical Taylor numbers in the small-gap approximation, and concluded that the error was probably about 30 of unknown sign in the cases $m = 1$ and $m = 2$. The results of table 4 show that this estimate was in fact pessimistic.

The main result, then, of the present work is to resolve any doubt about the validity of the answers obtained by their third-order expansion, and about the effect of the small-gap approximation. We see from table 4 that the general picture of possible instabilities of Taylor-vortex flow at values of T about 10 % above T_c is confirmed. Also the $m = 1$ mode appears to be marginally the most unstable, again in accordance with D. D. & S. and with an interpretation of the experiments of Schwarz, Springett & Donnelly (1964).

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Appendix. Non-linear terms

All the vectors $\mathbf{N}_j$ of equations (3.18) to (3.33) are obtained from $\mathbf{L}(\mathbf{U})\,\mathbf{U}$ of equation (2.10*a*). We outline the method of derivation and list the results (in a elementary fashion, since we feel this is clearer than a general formula).

Every function $\mathbf{u}_j(x)$ in equations (3.18) to (3.33) is a coefficient of

$$F^l(\tau)\, H^m(\tau)\, e^{ip\lambda\zeta + iqk\phi}$$

in the expansion (3.13) of $\mathbf{U}$. Only a certain selection of the values of l, m, p, q are needed in our problem, and we have labelled the functions we need arbitrarily in preference to using a clumsy suffix notation. The values of j associated with various values of l, m, p and q are listed in table 5.

To obtain the $\mathbf{N}_j$ we first find the partial differential equations for the $\mathbf{u}_{lmn}(x, \phi, \zeta)$ of expansion (3.13). These are obtained by picking out the coefficients of $F^l H^m H^n$. The first two are

$$(\partial/\partial x - \mathbf{A} - a_0 \mathbf{B})\, \mathbf{u}_{100} = 0, \tag{A 1}$$

$$(\partial/\partial x - \mathbf{A} - b_0 \mathbf{B})\, \mathbf{u}_{010} = 0. \tag{A 2}$$

The equation for $\mathbf{u}_{110}$ is

$$(\partial/\partial x - \mathbf{A} - \{a_0 + b_0\}\, \mathbf{B})\, \mathbf{u}_{110} = L(\mathbf{u}_{100})\, \mathbf{u}_{010} + \mathbf{L}(\mathbf{u}_{010})\, \mathbf{u}_{100}. \tag{A 3}$$

Now the functions $\mathbf{u}_{100}$ and $\mathbf{u}_{010}$ are given by (3.14*a*) and (3.14*b*). By substitution into (A 3) we see that $\mathbf{u}_{110}$ must have the form

$$\mathbf{u}_{110} = \mathbf{u}_5(x)\, e^{2i\lambda\zeta + ik\phi} + \mathbf{u}_6(x)\, e^{ik\phi} + \mathbf{v}_5(x)\, e^{-2i\lambda\zeta+ik\phi}. \quad \text{(A 4)}$$

The process may be continued, so that the form of each $\mathbf{u}_{lmn}$ is forced by the earlier functions. Each $\mathbf{u}_{lmn}$ is a sum of terms like $\mathbf{u}_j(x)\, e^{ip\lambda\zeta+iqk\phi}$ where the values of (p, q) are given by

$$(l+m+n, m-n),\ (l+m+n-2, m-n),\ \ldots,\ (-l-m-n, m-n). \quad \text{(A 5)}$$

The functions which we need for our purposes are listed in table 5.

In order to find the ordinary differential equations (3.13) to (3.33) we pick out the coefficients of $e^{ip\lambda\zeta+iqk\phi}$ in equations like (A 3). We find that the functions $\mathbf{v}_3$, $\mathbf{v}_5$, $\mathbf{v}_8$, $\mathbf{v}_{10}$ listed above satisfy differential equations which are simple transformations of those for correspondingly numbered $\mathbf{u}_j$, and hence that

$$\mathbf{v}_3 = \tilde{\mathbf{u}}_3, \quad \mathbf{v}_5 = -\hat{\mathbf{u}}_5, \quad \mathbf{v}_8 = \tilde{\mathbf{u}}_8, \quad \mathbf{v}_{10} = -\hat{\mathbf{u}}_{10}, \quad \text{(A 6)}$$

where the notation $\hat{\mathbf{u}}$ is defined in (3.12)†; and where we have used

$$\mathbf{v}_1 = \tilde{\mathbf{u}}_1 \quad \text{and} \quad \mathbf{v}_2 = -\hat{\mathbf{u}}_2 \quad \text{(A 7)}$$

as defined in (3.14*a*) and (3.14*b*).

l,	m,	p,	q	Function	l,	m,	p,	q	Function	l,	m,	p,	q	Function
1	0	1	0	$\mathbf{u}_1\,(=\mathbf{e}_0)$	1	1	2	1	$\mathbf{u}_5$	2	1	1	1	$\mathbf{u}_{10}$
1	0	−1	0	$\mathbf{v}_1\,(=\mathbf{g}_0)$	1	1	0	1	$\mathbf{u}_6$	2	1	−1	1	$\mathbf{v}_{10}$
0	1	1	1	$\mathbf{u}_2$	1	1	−2	1	$\mathbf{v}_5$	4	0	2	0	$\mathbf{u}_{11}$
0	1	−1	1	$\mathbf{v}_2$	3	0	3	0	$\mathbf{u}_7$	4	0	0	0	$\mathbf{u}_{12}$
2	0	2	0	$\mathbf{u}_3$	3	0	1	0	$\mathbf{u}_8$	3	1	2	1	$\mathbf{u}_{13}$
2	0	0	0	$\mathbf{u}_4$	3	0	−1	0	$\mathbf{v}_8$	3	1	0	1	$\mathbf{u}_{14}$
2	0	−2	0	$\mathbf{v}_3$	2	1	3	1	$\mathbf{u}_9$	5	0	1	0	$\mathbf{u}_{15}$
										4	1	1	1	$\mathbf{u}_{16}$

TABLE 5. The names of the functions which are coefficients of $F^l(\tau)\, H^m(\tau)\, e^{ip\lambda\zeta+iqk\phi}$ in expression (3·13) of $\mathbf{U}$

The matrix $\mathbf{M}\{\mathbf{f}(x), p, q\}$ is defined by

$$\left\{\begin{matrix}\mathbf{M}(\mathbf{f}, p, q) \text{ is obtained from } \mathbf{L}(\mathbf{U}) \text{ by replacing } \partial/\partial\zeta \text{ by} \\ ip\lambda,\ \partial/\partial\phi \text{ by } iqk \text{ and } \mathbf{U} \text{ by } \mathbf{f}.\end{matrix}\right\} \quad \text{(A 8)}$$

We define the vector $\mathbf{R}(\mathbf{f}, \mathbf{g})$ by:

$$\mathbf{R}(\mathbf{f}, \mathbf{g}) = \mathbf{M}(\mathbf{f}, p', q')\, \mathbf{g} + \mathbf{M}(\mathbf{g}, p'', q'')\, \mathbf{f}, \quad \text{(A 9)}$$

where p', q' are the harmonic numbers p, q associated with $\mathbf{g}$, and p'', q'' are those associated with $\mathbf{f}$ (see table 5).

† We note also that $\tilde{\mathbf{u}}_1 = \hat{\mathbf{u}}_1$, $\tilde{\mathbf{u}}_3 = \hat{\mathbf{u}}_3$, and $\mathbf{u}_8 = \hat{\mathbf{u}}_8$.

Then all the $\mathbf{N}_j$ are expressible as sums involving $\mathbf{R}(\mathbf{f},\mathbf{g})$ as follows:

$$\begin{aligned}
\mathbf{N}_3 &= \tfrac{1}{2}\mathbf{R}(\mathbf{u}_1,\mathbf{u}_1), \quad \mathbf{N}_4 = \mathbf{R}(\mathbf{u}_1,\mathbf{v}_1), \quad \mathbf{N}_5 = \mathbf{R}(\mathbf{u}_1,\mathbf{u}_2),\\
\mathbf{N}_6 &= \mathbf{R}(\mathbf{u}_2,\mathbf{v}_1)+\mathbf{R}(\mathbf{u}_1,\mathbf{v}_2), \quad \mathbf{N}_7 = \mathbf{R}(\mathbf{u}_3,\mathbf{u}_1),\\
\mathbf{N}_8 &= \mathbf{R}(\mathbf{u}_3,\mathbf{v}_1)+\mathbf{R}(\mathbf{u}_4,\mathbf{u}_1), \quad \mathbf{N}_9 = \mathbf{R}(\mathbf{u}_3,\mathbf{u}_2)+\mathbf{R}(\mathbf{u}_5,\mathbf{u}_1),\\
\mathbf{N}_{10} &= \mathbf{R}(\mathbf{u}_3,\mathbf{v}_2)+\mathbf{R}(\mathbf{u}_4,\mathbf{u}_2)+\mathbf{R}(\mathbf{u}_5,\mathbf{v}_1)+\mathbf{R}(\mathbf{u}_6,\mathbf{u}_1),\\
\mathbf{N}_{11} &= \mathbf{R}(\mathbf{u}_9,\mathbf{v}_1)+\mathbf{R}(\mathbf{u}_8,\mathbf{u}_1)+\mathbf{R}(\mathbf{u}_4,\mathbf{u}_3),\\
\mathbf{N}_{12} &= \mathbf{R}(\mathbf{u}_8,\mathbf{v}_1)+\mathbf{R}(\mathbf{u}_1,\mathbf{v}_8)+\mathbf{R}(\mathbf{u}_3,\mathbf{v}_3)+\tfrac{1}{2}\mathbf{R}(\mathbf{u}_4,\mathbf{u}_4),\\
\mathbf{N}_{13} &= \mathbf{R}(\mathbf{u}_7,\mathbf{v}_2)+\mathbf{R}(\mathbf{u}_8,\mathbf{u}_2)+\mathbf{R}(\mathbf{u}_9,\mathbf{v}_1)+\mathbf{R}(\mathbf{u}_{10},\mathbf{u}_1)+\mathbf{R}(\mathbf{u}_5,\mathbf{u}_4)+\mathbf{R}(\mathbf{u}_6,\mathbf{u}_3),\\
\mathbf{N}_{14} &= \mathbf{R}(\mathbf{u}_8,\mathbf{v}_2)+\mathbf{R}(\mathbf{u}_2,\mathbf{v}_8)+\mathbf{R}(\mathbf{u}_{10},\mathbf{v}_1)+\mathbf{R}(\mathbf{u}_1,\mathbf{v}_{10})+\mathbf{R}(\mathbf{u}_3,\mathbf{v}_5)+\mathbf{R}(\mathbf{u}_4,\mathbf{u}_6)\\
&\quad +\mathbf{R}(\mathbf{u}_5,\mathbf{v}_3),\\
\mathbf{N}_{15} &= \mathbf{R}(\mathbf{u}_{11},\mathbf{v}_1)+\mathbf{R}(\mathbf{u}_{12},\mathbf{u}_1)+\mathbf{R}(\mathbf{u}_7,\mathbf{v}_3)+\mathbf{R}(\mathbf{u}_8,\mathbf{u}_4)+\mathbf{R}(\mathbf{u}_3,\mathbf{v}_8),\\
\mathbf{N}_{16} &= \mathbf{R}(\mathbf{u}_{12},\mathbf{u}_2)+\mathbf{R}(\mathbf{u}_{11},\mathbf{v}_2)+\mathbf{R}(\mathbf{u}_7,\mathbf{v}_5)+\mathbf{R}(\mathbf{u}_8,\mathbf{u}_6)+\mathbf{R}(\mathbf{u}_5,\mathbf{v}_8)+\mathbf{R}(\mathbf{u}_3,\mathbf{v}_{10})\\
&\quad +\mathbf{R}(\mathbf{u}_{10},\mathbf{u}_4)+\mathbf{R}(\mathbf{u}_9,\mathbf{v}_3)+\mathbf{R}(\mathbf{u}_{13},\mathbf{v}_1)+\mathbf{R}(\mathbf{u}_{14},\mathbf{u}_1).
\end{aligned}$$

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